Interaction of a vortex ring with the free surface of an ideal fluid

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The interaction of a small vortex ring with the free surface of a perfect fluid is considered. In the frame of the point ring approximation, the asymptotic expression for the Fourier components of radiated surface waves is obtained in the case when the vortex ring comes from infinity and has both horizontal and vertical components of the velocity. The nonconservative corrections to the equations of motion of the ring, due to Cherenkov radiation, are derived.

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I. INTRODUCTION

The study of the interaction between vortex structures in a fluid and the free surface is important from both practical and theoretical points of view. In general, a detailed investigation of this problem is very hard. Even the theory of potential surface waves and the dynamics of vortices in an infinite space taken separately still have a lot of unsolved fundamental problems. Only the consideration of significantly simplified models can help us to understand the processes which take place in the combined system.

In many cases it is possible to neglect the compressibility of the fluid as well as the energy dissipation. Therefore, the model of ideal homogeneous incompressible fluid is very useful for hydrodynamics. Because of the conservative nature of this model, the application of the well developed apparatus of Hamiltonian dynamics becomes possible [1,2]. An example of an effective use of the Hamiltonian formalism in hydrodynamics is the introduction of canonical variables for investigations of potential flows of perfect fluids with a free boundary. Zakharov showed at the end of the 1960s [3] that the surface shape $z=\eta(x,y,t)$ and the value of the velocity potential $\psi(x,y,t)$ on the surface can be considered as generalized coordinate and momentum, respectively.

It is important to note that a variational formulation of Hamiltonian dynamics in many cases allows us to obtain good finite-dimensional approximations which reflect the main features of the behavior of the original system. There are several possibilities for a parametrization of nonpotential flows of perfect fluid by some variables with dynamics determined by a variational principle. All are based on the conservation of the topological characteristics of vortex lines in ideal fluid flows which follows from the freezing-in of the vorticity field $\mathbf{\Omega}(\mathbf{r},t)=\mathrm{curl}\,\mathbf{v}(\mathbf{r},t)$. In particular, this is the representation of the vorticity by Clebsch canonical variables λ and μ [4,2],

$$\mathbf{\Omega}(\mathbf{r},t)=[\nabla\lambda\times\nabla\mu].$$

However, the Clebsch representation can only describe flows with a trivial topology (see, e.g., [5]). It cannot describe flows with linked vortex lines. Besides, the variables λ and μ are not suitable for the study of localized vortex structures such as vortex filaments. In such cases it is more convenient to use the parametrization of vorticity in terms of vortex lines and consider the motion of these lines [6,7], even if the global definition of canonically conjugated variables is impossible due to topological reasons.

This approach is used in the present paper to describe the interaction of deep (or small) vortex rings of almost ideal shape in the perfect fluid with the free surface. Besides a theoretical interest, the solution of this mathematical problem gives a qualitative explanation of some simple laboratory experiments with real vortex rings in water. In the case under consideration, the main interaction of the vortex rings with the surface can be described as the dipole-dipole interaction between "point" vortex rings and their "images." Moving rings interact with the surface waves, leading to radiation due to the Cherenkov effect. Deep rings disturb the surface weakly, so the influence of the surface can be taken into account as some small corrections in the equations of motion for the parameters of the rings.

In Sec. II, we discuss briefly general properties of vortex line dynamics, which follow from the freezing-in of the vorticity field. In Sec. III, possible simplifications of the model are made and the point ring approximation is introduced. In Sec. IV, the interaction of the ring with its image is considered. In Sec. V, we calculate the Fourier components of Cherenkov surface waves radiated by a moving vortex ring and determine the nonconservative corrections caused by the interaction with the surface for the vortex ring equations of motion.

II. VORTEX LINE MOTION IN PERFECT FLUID

It is a well known fact that the freezing-in of the vorticity lines follows from the Euler equation for ideal fluid motion,

$$\mathbf{\Omega}_t=\mathrm{curl}[\mathbf{v}\times\mathbf{\Omega}],\quad \mathbf{v}=\mathrm{curl}^{-1}\,\mathbf{\Omega}.$$

Vortex lines are transported by the flow [1,4,8]. They do not appear or disappear, nor do they intersect in the process of motion. This property of perfect fluid flows is general for all Hamiltonian systems of the hydrodynamic type. For simplicity, let us consider temporally the incompressible fluid without the free surface in infinite space. The dynamics of the system is specified by a basic Lagrangian $L[\mathbf{v}]$, which is a functional of the solenoidal velocity field. The relations be-

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tween the velocity $\mathbf{v}$, the generalized vorticity $\mathbf{\Omega}$, the basic Lagrangian $L[\mathbf{v}]$, and the Hamiltonian $\mathcal{H}[\mathbf{\Omega}]$ are the following [9]:

$$\mathbf{\Omega}=\mathrm{curl}(\delta L/\delta \mathbf{v}) \Rightarrow \mathbf{v}=\mathbf{v}[\mathbf{\Omega}], \quad (2.1)$$

$$\mathcal{H}[\mathbf{\Omega}]=\left(\int \mathbf{v}\cdot(\delta L/\delta \mathbf{v})d^3\mathbf{r}-L[\mathbf{v}]\right)\Bigg|_{\mathbf{v}=\mathbf{v}[\mathbf{\Omega}]}, \quad (2.2)$$

$$\mathbf{v}=\mathrm{curl}(\delta\mathcal{H}/\delta\mathbf{\Omega}). \quad (2.3)$$

For ordinary Eulerian ideal hydrodynamics in infinite space, the basic Lagrangian is

$$L_E[\mathbf{v}]=\int \frac{\mathbf{v}^2}{2}d\mathbf{r} \Rightarrow \mathbf{\Omega}_E=\mathrm{curl}\,\mathbf{v}.$$

The Hamiltonian in this case coincides with the kinetic energy of the fluid and in terms of the vorticity field it reads

$$\mathcal{H}_E[\mathbf{\Omega}]=-\tfrac{1}{2}\int \mathbf{\Omega}\Delta^{-1}\mathbf{\Omega}d\mathbf{r}=\int\int \frac{\mathbf{\Omega}(\mathbf{r}_1)\cdot\mathbf{\Omega}(\mathbf{r}_2)}{8\pi|\mathbf{r}_1-\mathbf{r}_2|}d\mathbf{r}_1 d\mathbf{r}_2,$$

where Δ^{-1} is the inverse Laplace operator.

Another example is the basic Lagrangian of electron magnetohydrodynamics (EMHD) which takes into account the magnetic field created by the current of electron fluid through the motionless ion fluid,

$$L_{\mathrm{EMHD}}[\mathbf{v}]=\tfrac{1}{2}\int \mathbf{v}(1-\Delta^{-1})\mathbf{v}d\mathbf{r},$$

$$\mathbf{\Omega}_{\mathrm{EMHD}}=\mathrm{curl}(1-\Delta^{-1})\mathbf{v}.$$

The Hamiltonian of EMHD is

$$\mathcal{H}_{\mathrm{EMHD}}[\mathbf{\Omega}]=\frac{1}{8\pi}\int\int \frac{e^{-|\mathbf{r}_1-\mathbf{r}_2|}}{|\mathbf{r}_1-\mathbf{r}_2|}\mathbf{\Omega}(\mathbf{r}_1)\cdot\mathbf{\Omega}(\mathbf{r}_2)d\mathbf{r}_1 d\mathbf{r}_2.$$

The second example shows that the relation between the velocity and the vorticity can be more complex than in usual hydrodynamics. In the general case, a Hamiltonian of some hydrodynamical system is not a necessary quadratic functional of the vorticity (see the examples of integrable Hamiltonians in [11]).

The equation of motion for the generalized vorticity is

$$\mathbf{\Omega}_t=\mathrm{curl}[\mathrm{curl}(\delta\mathcal{H}/\delta\mathbf{\Omega})\times\mathbf{\Omega}]. \quad (2.4)$$

This equation corresponds to the transport of frozen-in vortex lines by the velocity field. In this process, all topological invariants [10] of the vorticity field are conserved. The conservation of the topology can be expressed by the following relation [7]:

$$\mathbf{\Omega}(\mathbf{r},t)=\int \delta(\mathbf{r}-\mathbf{R}(\mathbf{a},t))(\mathbf{\Omega}_0(\mathbf{a})\nabla_{\mathbf{a}})\mathbf{R}(\mathbf{a},t)d\mathbf{a}$$
$$=\frac{(\mathbf{\Omega}_0(\mathbf{a})\nabla_{\mathbf{a}})\mathbf{R}(\mathbf{a},t)}{\det\|\partial\mathbf{R}/\partial\mathbf{a}\|}\Bigg|_{\mathbf{a}=\mathbf{a}(\mathbf{r},t)}, \quad (2.5)$$

where the mapping $\mathbf{R}(\mathbf{a},t)$ describes the deformation of lines of some initial solenoidal field $\mathbf{\Omega}_0(\mathbf{r})$. Here $\mathbf{a}(\mathbf{r},t)$ is the inverse mapping with respect to $\mathbf{R}(\mathbf{a},t)$. The direction of the vector $\mathbf{b}$,

$$\mathbf{b}(\mathbf{a},t)=(\mathbf{\Omega}_0(\mathbf{a})\nabla_{\mathbf{a}})\mathbf{R}(\mathbf{a},t), \quad (2.6)$$

coincides with the direction of the vorticity field at the point $\mathbf{R}(\mathbf{a},t)$. The equation of motion for the mapping $\mathbf{R}(\mathbf{a},t)$ can be obtained with the help of the relation

$$\mathbf{\Omega}_t(\mathbf{r},t)=\mathrm{curl}_{\mathbf{r}}\int \delta(\mathbf{r}-\mathbf{R}(\mathbf{a},t))[\mathbf{R}_t(\mathbf{a},t)\times\mathbf{b}(\mathbf{a},t)]d\mathbf{a}, \quad (2.7)$$

which immediately follows from Eq. (2.5). The substitution of Eq. (2.7) into the equation of motion (2.4) gives [11]

$$\mathrm{curl}_{\mathbf{r}}\left(\frac{\mathbf{b}(\mathbf{a},t)\times[\mathbf{R}_t(\mathbf{a},t)-\mathbf{v}(\mathbf{R},t)]}{\det\|\partial\mathbf{R}/\partial\mathbf{a}\|}\right)=0.$$

One can solve this equation by eliminating the $\mathrm{curl}_{\mathbf{r}}$ operator. Using the general relationship between variational derivatives of some functional $F[\mathbf{\Omega}]$,

$$\left[\mathbf{b}\times\mathrm{curl}\left(\frac{\delta F}{\delta\mathbf{\Omega}(\mathbf{R})}\right)\right]=\frac{\delta F}{\delta\mathbf{R}(\mathbf{a})}\Bigg|_{\mathbf{\Omega}_0}, \quad (2.8)$$

it is possible to represent the equation of motion for $\mathbf{R}(\mathbf{a},t)$ as follows:

$$[(\mathbf{\Omega}_0(\mathbf{a})\nabla_{\mathbf{a}})]\times\mathbf{R}(\mathbf{a})\mathbf{R}_t(\mathbf{a})\}=\frac{\delta\mathcal{H}[\mathbf{\Omega}[\mathbf{R}]]}{\delta\mathbf{R}(\mathbf{a})}\Bigg|_{\mathbf{\Omega}_0}. \quad (2.9)$$

It is not difficult to check now that the dynamics of the vorticity field with topological properties defined by $\mathbf{\Omega}_0$ in the infinite space is equivalent to the requirement of an extremum of the action ($\delta S=\delta\int\mathcal{L}_{\mathbf{\Omega}_0}dt=0$), where the Lagrangian is

$$\mathcal{L}_{\mathbf{\Omega}_0}=\int ([\mathbf{R}_t(\mathbf{a})\times\mathbf{D}(\mathbf{R}(\mathbf{a}))]\cdot(\mathbf{\Omega}_0(\mathbf{a})\nabla_{\mathbf{a}})\mathbf{R}(\mathbf{a}))d\mathbf{a}$$
$$-\mathcal{H}[\mathbf{\Omega}[\mathbf{R}]]. \quad (2.10)$$

Here the vector function $\mathbf{D}(\mathbf{R})$ must have the unit divergency

$$(\nabla_{\mathbf{R}}\cdot\mathbf{D}(\mathbf{R}))=1 \quad (2.11)$$

and it is defined up to the addition of an arbitrary solenoidal vector function. It should be mentioned that for the case of ordinary Eulerian incompressible hydrodynamics, the variational principle (2.10) with the particular form $\mathbf{D}=\mathbf{R}/3$ was proved in the works of Berdichevsky [6]. For general Hamiltonian systems of the hydrodynamic type, this principle (also with $\mathbf{D}=\mathbf{R}/3$) was formulated in the paper of Kuznetsov and Ruban [7]. Furthermore, in the present paper we will use the above-mentioned choice of $\mathbf{D}(\mathbf{R})$.

The topological properties of the vorticity field can be very complicated. For instance, a single vortex line can fill a three-dimensional domain. In a more simple case, each vortex line forms a two-dimensional vortex surface. In the simplest case, when all vortex lines are closed it is possible to choose new curvilinear coordinates ν_1,ν_2,ξ in $\mathbf{a}$ space such that Eq. (2.5) can be written in a simple form,

$$\mathbf{\Omega}(\mathbf{r},t)=\int_{\mathcal{N}}d^2\nu\oint\delta(\mathbf{r}-\mathbf{R}(\nu,\xi,t))\mathbf{R}_\xi d\xi. \quad (2.12)$$

Here ν is the label of a line lying on a fixed two-dimensional manifold $\mathcal{N}$, and ξ is some parameter along the line. Without loss of generality, we omit a given function $\gamma_0(\nu_1,\nu_2)$ in front of $d^2\nu$ because it can always be made equal to unity by a redefinition of ν_1 and ν_2. It is clear that there is a gauge freedom in the definition of ν and ξ in Eq. (2.12). This freedom is connected with the possibility of changing the longitudinal parameter

$$\xi=\xi(\tilde{\xi},\nu,t)$$

and also with the relabeling of ν,

$$\nu=\nu(\tilde{\nu},t), \quad \frac{\partial(\nu_1,\nu_2)}{\partial(\tilde{\nu}_1,\tilde{\nu}_2)}=1. \tag{2.13}$$

Now we consider again the ordinary perfect fluid with a free surface. To describe the flow entirely, it is sufficient to specify the vorticity field $\mathbf{\Omega}(\mathbf{r},t)$ and the motion of the free surface. Thus, we can use the shape $\mathbf{R}(\nu,\xi,t)$ of the vortex lines as a new dynamic object instead of $\mathbf{\Omega}(\mathbf{r},t)$. It is important to note that in the presence of the free surface, the equations of motion for $\mathbf{R}(\nu,\xi,t)$ follow from a variational principle as in the case of infinite space. It has been shown [12] that the Lagrangian for a perfect fluid, with vortices in its bulk and with a free surface, can be written in the form

$$\mathcal{L}=\int_{\mathcal{N}}\frac{d^2\nu}{3}\oint([\mathbf{R}_t\times\mathbf{R}]\cdot\mathbf{R}_\xi)d\xi+\int\Psi\,\eta_t d\mathbf{r}_\perp-\mathcal{H}[\mathbf{R},\Psi,\eta]. \tag{2.14}$$

The functions $\Psi(\mathbf{r}_\perp,t)$ and $\eta(\mathbf{r}_\perp,t)$ are the surface degrees of freedom for the system. Ψ is the boundary value of the total velocity potential, which includes the part from vortices inside the fluid, and η is the deviation of the surface from the horizontal plane. This formulation supposes that vortex lines do not intersect the surface anywhere. In the present paper, only this case is considered.

The Hamiltonian $\mathcal{H}$ in Eq. (2.14) is merely the total energy of the system expressed in terms of $[\mathbf{R},\Psi,\eta]$.

Variation with respect to $\mathbf{R}(\nu,\xi,t)$ of the action defined by the Lagrangian (2.14) gives the equation of motion for vortex lines in the form

$$[\mathbf{R}_\xi\times\mathbf{R}_t]=\frac{\delta\mathcal{H}[\mathbf{\Omega}[\mathbf{R}],\Psi,\eta]}{\delta\mathbf{R}}. \tag{2.15}$$

This equation determines only the transversal component of $\mathbf{R}_t$, which coincides with the transversal component of the actual solenoidal velocity field. The possibility of solving Eq. (2.15) with respect to the time derivative $\mathbf{R}_t$ is closely connected with the special gauge-invariant nature of the $\mathcal{H}[\mathbf{R}]$ dependence, which results in

$$\frac{\delta\mathcal{H}}{\delta\mathbf{R}}\cdot\mathbf{R}_\xi\equiv 0.$$

The tangential component of $\mathbf{R}_t$ with respect to vorticity direction can be taken arbitrary. This property is in accordance with the longitudinal gauge freedom. The vorticity dynamics does not depend on the choice of the tangential component.

Generally speaking, only the local introduction of canonical variables for curve dynamics is possible. For instance, a piece of the curve can be parametrized by one of the three Cartesian coordinates,

$$\mathbf{R}=(X(z,t),Y(z,t),z).$$

In this case, the functions $X(z,t)$ and $Y(z,t)$ are canonically conjugated variables. Another example is the parametrization in cylindrical coordinates, where variables $Z(\theta,t)$ and $(1/2)R^2(\theta,t)$ are canonically conjugated. Curves with complicated topological properties need a general gauge-free description by means of a parameter ξ.

It should be mentioned for clarity that the conservation of all vortex tube volumes, reflecting the incompressibility of the fluid, is not the constraint in this formalism. It is a consequence of the symmetry of the Lagrangian (2.14) with respect to the relabeling (2.13) $\nu\to\tilde{\nu}$ [6,9]. Volume conservation follows from that symmetry in accordance with Noether's theorem. To prove this statement, we should consider such a subset of relabelings which forms a one-parameter group of transformations of the dynamical variables,

$$\mathbf{R}(\nu_1,\nu_2,\xi)\to\mathbf{R}^\tau(\nu_1,\nu_2,\xi).$$

For small values of the group parameter, τ, the relabelings $\nu(\tilde{\nu},\tau)$ are specified by a function of two variables $T(\nu_1,\nu_2)$ (with zero value on the boundary $\partial\mathcal{N}$) so that the corresponding transformations are

$$\mathbf{R}_T^\tau(\nu_1,\nu_2,\xi)=\mathbf{R}\left(\nu_1-\tau\frac{\partial T}{\partial\nu_2}+O(\tau^2),\ \ \nu_2+\tau\frac{\partial T}{\partial\nu_1}+O(\tau^2),\ \ \xi\right). \tag{2.16}$$

Due to Noether's theorem, the following quantity is an integral of motion [13]:

$$I_T=\int_{\mathcal{N}}d^2\nu\oint\frac{\delta\mathcal{L}}{\delta\mathbf{R}_t}\cdot\frac{\partial\mathbf{R}_T^\tau}{\partial\tau}\bigg|_{\tau=0}d\xi$$
$$=\frac{1}{3}\int_{\mathcal{N}}d^2\nu\oint[\mathbf{R}\times\mathbf{R}_\xi]\cdot(\mathbf{R}_2T_1-\mathbf{R}_1T_2)d\xi.$$

After simple integrations by parts, the preceding expression takes the form [6]

$$I_T=\int_{\mathcal{N}}d^2\nu\oint T(\nu_1,\nu_2)([\mathbf{R}_1\times\mathbf{R}_2]\cdot\mathbf{R}_\xi)d\xi$$
$$=\int_{\mathcal{N}}T(\nu_1,\nu_2)\mathcal{V}(\nu_1,\nu_2,t)d^2\nu, \tag{2.17}$$

where $\mathcal{V}(\nu_1,\nu_2,t)d^2\nu$ is the volume of an infinitely thin vortex tube with cross section $d^2\nu$. It is obvious that actually the function $\mathcal{V}$ does not depend on time t because the function $T(\nu_1,\nu_2)$ is arbitrary.

If vortex lines are not closed but form a family of enclosed tori, then the relabeling freedom is less rich. In that case, one can obtain in a similar way the conservation laws for volumes inside closed vortex surfaces. Noether's theorem gives integrals of motion which depend on an arbitrary function $S(\zeta)$ of one variable, where ζ is the label of the tori.

In the case of complicated topology of the vorticity field, there is a correspondence between each globally defined vortex surface, if any exists, and the conservation of the volume inside that vortex surface.

It is easy to check that if instead of the old Hamiltonian $\mathcal{H}[\mathbf{\Omega}[\mathbf{R}]]$ we consider the new Hamiltonian

$$\tilde{\mathcal{H}}[\mathbf{R}]=\mathcal{H}[\mathbf{\Omega}[\mathbf{R}]]+I_T,$$

then the dynamics of the vorticity $\mathbf{\Omega}(\mathbf{r},t)$ will remain the same. The term I_T produces just a relabeling (2.13) of ν in the equation of motion for $\mathbf{R}$. Each stationary solution $\mathbf{R}_{\text{stat}}(\nu,\xi)$ with this new Hamiltonian and some particular choice of $T(\nu_1,\nu_2)$ describes a stationary flow of the fluid. The stability of such a flow depends on a definiteness of the sign of the second variation $\delta^2\tilde{\mathcal{H}}[\mathbf{R}_{\text{stat}}]$ near the stationary distribution $\mathbf{R}_{\text{stat}}$ of the vortex lines.

III. POINT RING APPROXIMATION

In a general case, an analysis of the dynamics defined by the Lagrangian (2.14) is much too complicated. We do not even have the exact expression for the Hamiltonian $\mathcal{H}[\mathbf{R},\Psi,\eta]$ because it requires the explicit knowledge of the solution of the Laplace equation with a boundary value assigned on a nonflat surface. Another reason is the very high nonlinearity of the problem.

In this paper, we consider some limits where it is possible to simplify the system significantly. Namely, we will suppose that the vorticity is concentrated in several very thin vortex rings of almost ideal shape. For a solitary ring, the perfect shape is stable for a wide range of vorticity distributions through the cross section. This shape provides an extremum of the energy for given values of the volumes of vortex tubes and for a fixed momentum of the ring. As already mentioned, volume conservation follows from Noether's theorem. Therefore, some of these quantities (those of which are produced by the subset of commuting transformations) can be considered as canonical momenta. Corresponding cyclical coordinates describe the relabeling (2.13) of the line markers, which does not change the vorticity field. Actually, these degrees of freedom take into account a rotation around the central line of the tube. This line represents the mean shape of the ring and we are interested in how it behaves in time. For our analysis, we do not need the explicit values of cyclical coordinates, but only the conserved volumes as parameters in the Lagrangian.

A possible situation is when a typical time of the interaction with the surface and with other rings is much larger than the largest period of oscillations corresponding to deviations of the ring shape from the perfect one. Under this condition, excitations of all (noncyclical) internal degrees of freedom are always small, and a variational ansatz completely disregarding them reflects the behavior of the system adequately. The circulations

$$\Gamma_n=\int_{\mathcal{N}_n} d^2\nu$$

of the velocity for each ring do not depend on time. A perfect ring is described by the coordinate $\mathbf{R}_n$ of the center and by the vector $\mathbf{P}_n=\Gamma_n\mathbf{S}_n$, where $\mathbf{S}_n$ is an oriented area of the ring. We use in this work the Cartesian system of coordinates (x,y,z), so that the vertical coordinate is z and the unperturbed surface is at $z=0$. The corresponding components of the vectors $\mathbf{R}_n$ and $\mathbf{P}_n$ are

$$\mathbf{R}_n=(X_n,Y_n,Z_n),\quad \mathbf{P}_n=(P_{xn},P_{yn},P_{zn}).$$

It is easy to verify that the vectors $\mathbf{P}_n$ are canonically conjugated momenta for the coordinates $\mathbf{R}_n$. To verify that, we can parametrize the shape of each vortex line in the following manner:

$$\mathbf{R}(\xi,t)=\sum_{m=-M}^{M}\mathbf{r}_m(t)e^{im\xi},\quad \mathbf{r}_{-m}=\overline{\mathbf{r}_m}. \tag{3.1}$$

Here $\mathbf{r}_m(t)$ are complex vectors. Substituting this into the first term of the Lagrangian (2.14) gives

$$\begin{aligned}\frac{1}{3}\oint([\mathbf{R}_t\times\mathbf{R}]\mathbf{R}_\xi)d\xi &=2\pi i\dot{\mathbf{r}}_0([\mathbf{r}_{-1}\times\mathbf{r}_1]+2[\mathbf{r}_{-2}\times\mathbf{r}_2]+\cdots)+\frac{d\{\cdots\}}{dt}\\ &+2\pi i(\dot{\mathbf{r}}_{-1}[\mathbf{r}_{-1}\times\mathbf{r}_2]-\dot{\mathbf{r}}_1[\mathbf{r}_1\times\mathbf{r}_{-2}])+\cdots .\end{aligned} \tag{3.2}$$

If we neglect the internal degrees of freedom which describe deviations of the ring from the ideal shape

$$(\mathbf{r}_{-1})^2=(\mathbf{r}_1)^2=0,\quad \mathbf{r}_2=\mathbf{r}_{-2}=0,\ldots,$$

then the previous statement about canonically conjugated variables becomes obvious:

$$\mathbf{R}_n=\mathbf{r}_{0n},\quad \mathbf{P}_n=2\pi\Gamma_n\cdot i[\mathbf{r}_{-1n}\times\mathbf{r}_{1n}]. \tag{3.3}$$

Such an approximation is valid only in the limit when sizes of rings are small in comparison with the distances to the surface and the distances between different rings,

$$\sqrt{\frac{P_n}{\Gamma_n}}\ll|Z_n|,|\mathbf{R}_n-\mathbf{R}_l|,\quad l\neq n. \tag{3.4}$$

These conditions are necessary for ensuring that the excitations of all internal degrees of freedom are small. Obviously, this is not true when a ring approaches the surface. In that case, one should take into account also the internal degrees of freedom for the vortex lines.

The inequalities (3.4) also imply that vortex rings in the limit under consideration are similar to point magnet dipoles. This analogy is useful for calculation of the Hamiltonian for interacting rings. In the main approximation, we may restrict the analysis by taking into account the dipole-dipole interaction only.

It should be mentioned that in some papers (see, e.g., [14] and references in that book) the discrete variables identical to $\mathbf{R}_n$ and $\mathbf{P}_n$ are derived in a different way and referred to as the vortex magnetization variables.

In the expression for the Hamiltonian, several simplifications can be made. Let us recall that for each moment of time it is possible to decompose the velocity field into two components,

$$\mathbf{v}=\mathbf{V}_0+\boldsymbol{\nabla}\phi. \tag{3.5}$$

Here the field $\mathbf{V}_0$ satisfies the following conditions:

$$(\boldsymbol{\nabla}\cdot\mathbf{V}_0)=0, \quad \mathrm{curl}\,\mathbf{V}_0=\boldsymbol{\Omega}, \quad (\mathbf{n}\cdot\mathbf{V}_0)|_{z=\eta}=0.$$

The boundary value of the surface wave potential $\phi(\mathbf{r})$ is $\psi(\mathbf{r}_\perp)$. In accordance with these conditions, the kinetic energy is decomposed into two parts and the Hamiltonian of the fluid takes the form

$$\mathcal{H}=\tfrac{1}{2}\int_{z<\eta}\mathbf{V}_0^2 d^3\mathbf{r}+\tfrac{1}{2}\int \psi(\boldsymbol{\nabla}\phi\cdot d\mathbf{S})+\frac{g}{2}\int \eta^2 d\,\mathbf{r}_\perp\,. \tag{3.6}$$

The last term in this expression is the potential energy of the fluid in the gravitational field. If all vortex rings are far away from the surface, then its deviation from the horizontal plane is small,

$$|\boldsymbol{\nabla}\eta|\ll 1, \quad |\eta|\ll|Z_n|. \tag{3.7}$$

Therefore, in the main approximation the energy of dipole interaction with the surface can be described with the help of so-called ''images.'' The images are vortex rings with circulations Γ_n and parameters

$$\mathbf{R}_n^*=(X_n,Y_n,-Z_n), \quad \mathbf{P}_n^*=(P_{xn},P_{yn},-P_{zn}). \tag{3.8}$$

The kinetic energy for the system of point rings and their images is the sum of the self-energies of rings and the dipole-dipole interaction between them. The expression for the kinetic energy of small-amplitude surface waves employs the operator $\hat{k}$, which multiplies Fourier components of a function by the absolute value k of a two-dimensional wave vector $\mathbf{k}$. So the actual Hamiltonian $\mathcal{H}$ is approximately equal to the simplified Hamiltonian $\tilde{\mathcal{H}}$,

$$\begin{aligned}\mathcal{H}\approx\tilde{\mathcal{H}}=&\sum_n \mathcal{E}_n(P_n)+\frac{1}{2}\int(\psi\hat{k}\psi+g\eta^2)d\mathbf{r}_\perp+\frac{1}{8\pi}\\ &\times\sum_{ln}\frac{3(\mathbf{R}_{nl*}\cdot\mathbf{P}_n)(\mathbf{R}_{nl*}\cdot\mathbf{P}_l^*)-|\mathbf{R}_{nl*}|^2(\mathbf{P}_n\cdot\mathbf{P}_l^*)}{|\mathbf{R}_{nl*}|^5}+\frac{1}{8\pi}\\ &\times\sum_{l\neq n}\frac{3(\mathbf{R}_{nl}\cdot\mathbf{P}_n)(\mathbf{R}_{nl}\cdot\mathbf{P}_l)-|\mathbf{R}_{nl}|^2(\mathbf{P}_n\cdot\mathbf{P}_l)}{|\mathbf{R}_{nl}|^5},\end{aligned} \tag{3.9}$$

where

$$\mathbf{R}_{nl}=\mathbf{R}_n-\mathbf{R}_l\,, \quad \mathbf{R}_{nl*}=\mathbf{R}_n-\mathbf{R}_l^*\,. \tag{3.10}$$

With logarithmic accuracy, the self-energy of a thin vortex ring is given by the expression

$$\mathcal{E}_n(P_n)\approx\frac{\Gamma_n^2}{2}\sqrt{\frac{P_n}{\pi\Gamma_n}}\ln\left(\frac{(P_n/\Gamma_n)^{3/4}}{A_n^{1/2}}\right), \tag{3.11}$$

where the small constant A_n is proportional to the conserved volume of the vortex tube forming the ring. This expression can easily be derived if we take into account that the main contribution to the energy is from the vicinity of the tube where the velocity field is approximately the same as near a straight vortex tube. The logarithmic integral should then be taken between the limits from the thickness of the tube to the radius of the ring.

In the relation $\Psi=\Phi_0+\psi$, the potential Φ_0 is approximately equal to the potential created on the flat surface by the dipoles and their images,

$$\Phi_0(\mathbf{r}_\perp)\approx\Phi(\mathbf{r}_\perp)=-\frac{1}{2\pi}\sum_n\frac{(\mathbf{P}_n\cdot(\mathbf{r}_\perp-\mathbf{R}_n))}{|\mathbf{r}_\perp-\mathbf{R}_n|^3}. \tag{3.12}$$

In this way we arrive at the following simplified system describing the interaction of point vortex rings with the free surface:

$$\tilde{\mathcal{L}}=\sum_n \dot{\mathbf{R}}_n\mathbf{P}_n+\int\dot{\eta}(\psi+\Phi)d^2\mathbf{r}_\perp-\tilde{\mathcal{H}}[\{\mathbf{R}_n,\mathbf{P}_n\},\eta,\psi]. \tag{3.13}$$

It should be noted that due to the condition (3.4), the maximum value of the velocity V_0 on the surface is much less than the typical velocities of the vortex rings,

$$\frac{P_n}{Z_n^3}\ll\frac{\Gamma_n^{3/2}}{P_n^{1/2}}.$$

Therefore, the term $V_0^2/2$ in the Bernoulli equation

$$\Psi_t+V_0^2/2+g\eta+(\text{small corrections})=0$$

is small in comparison with the term Ψ_t. The Lagrangian (3.13) is in accordance with this fact because it does not take into account terms like $(1/2)\int V_0^2\eta\,d^2\mathbf{r}_\perp$ in the Hamiltonian expansion.

IV. INTERACTION OF THE VORTEX RING WITH ITS IMAGE

Now let us consider for simplicity the case of a single ring. It is shown in the next section that for a sufficiently deep ring, the interaction with its image is much stronger than the interaction with the surface waves. So it is interesting to examine the motion of the ring neglecting the surface deviation. In this case we have the integrable Hamiltonian for the system with two degrees of freedom (motion is in a vertical plane),

$$H=\frac{1}{64\pi}\left(\alpha(P_x^2+P_z^2)^{1/4}-\frac{2P_z^2+P_x^2}{|Z|^3}\right), \tag{4.1}$$

where $\alpha\approx$const. The system has integrals of motion,

$$P_x=p=\text{const}, \quad H=E=\text{const},$$

so it is useful to consider the level lines of the energy function in the left (Z,P_z)-half-plane taking P_x as the parameter. If one will draw the sketch of level lines of the function $H(Z,P_z)$, Eq. (4.1), then he will immediately distinguish three regions of qualitatively different behavior of the ring in that part of this half-plane where our approximation is valid [see Eq. (3.4)]. In the upper region, the phase trajectories come from infinitely large negative Z where they have a finite positive value of P_z. In the course of motion, P_z increases. This behavior corresponds to the case when the ring approaches the surface. Due to the symmetry of the Hamiltonian (4.1), there is the symmetric lower region, where the vortex ring moves away from the surface, and there is the middle region, where P_z changes sign from negative to positive at a finite value of Z. This is the region of the finite motion.

In all three cases, the track of the vortex ring bends toward the surface, i.e., the ring is ''attracted'' by the surface.

V. CHERENKOV INTERACTION OF A VORTEX RING WITH SURFACE WAVES

When the ring is not very far from the surface and not very slow, the interaction with the surface waves becomes significant. Let us consider the effect of Cherenkov radiation of surface waves by a vortex ring which moves from infinity to the surface. This case is the most definite from the viewpoint of the choice of initial conditions. We suppose that the deviation of the free surface from the horizontal plane $z=0$ is zero at $t\to-\infty$, and we are interested in the asymptotic behavior of fields η and ψ at large negative t. In this situation we can neglect the interaction of the ring with its image in comparison with the self-energy and concentrate our attention on interaction with surface waves only.

The ring moves in the (x,z) plane with an almost constant velocity. In the main approximation the position $\mathbf{R}$ of the vortex ring is given by the relations

$$\mathbf{R}\approx\mathbf{C}t,\quad \mathbf{C}=\frac{\partial\mathcal{E}(\mathbf{P})}{\partial\mathbf{P}}=(C_x,0,C_z)\sim\frac{\mathbf{P}}{P^{3/2}}, \tag{5.1}$$

$$C_x>0,\quad C_z>0,\quad t<0.$$

The equations of motion for the Fourier components of η and ψ follow from the Lagrangian (3.13),

$$\dot{\eta}_{\mathbf{k}}=k\psi_{\mathbf{k}},\quad \dot{\psi}_{\mathbf{k}}+g\,\eta_{\mathbf{k}}=-\dot{\Phi}_{\mathbf{k}}. \tag{5.2}$$

[The equations for the surface waves are the same in the case when a rigid body moves deeply under the surface of an ideal fluid. The difference is in the relations $\mathbf{C}(\mathbf{P})$ between momenta and velocities. As against the Eq. (5.1), for a rigid body $C\sim P$. The equations of motion for $\mathbf{P}$ also differ. Nevertheless, if the velocity of a vortex ring or a rigid body is almost constant, as it is assumed here, then in the main approximation it is not possible to distinguish what approaches the water surface, rigid body, or vortex ring by measuring the corresponding surface pattern. The difference in a temporal behavior of the patterns appears only when the velocity is changed significantly by an interaction in the course of motion. So, in principle, the situation can be resolved. Probably, an appropriate analysis should take into account also the effects of the viscosity. This interesting problem is not considered in the present paper.]

Eliminating $\eta_{\mathbf{k}}$, we obtain an equation for $\psi_{\mathbf{k}}$,

$$\ddot{\psi}_{\mathbf{k}}+gk\psi_{\mathbf{k}}=-\ddot{\Phi}_{\mathbf{k}}, \tag{5.3}$$

where $\Phi_{\mathbf{k}}$ is the Fourier transform of the function $\Phi(\mathbf{r}_\perp)$. Simple calculations give

$$\Phi_{\mathbf{k}}=\frac{e^{-ik_xX}}{2\pi}\int\frac{P_zZ-P_xx}{\sqrt{(x^2+y^2+Z^2)^3}}e^{-i(k_xx+k_yy)}dx\,dy$$
$$=-\frac{e^{-ik_xX}}{2\pi}\left(P_zD(k|Z|)+i\frac{P_x}{|Z|}\frac{\partial}{\partial k_x}D(k|Z|)\right), \tag{5.4}$$

where

$$D(q)=\int\frac{e^{-iq\alpha}d\alpha\,d\beta}{\sqrt{(\alpha^2+\beta^2+1)^3}}=2\pi e^{-|q|}. \tag{5.5}$$

Finally, we have for $\Phi_{\mathbf{k}}$,

$$\Phi_{\mathbf{k}}=\left(\frac{iP_xk_x}{k}-P_z\right)e^{-k|Z|-ik_xX}=\left(\frac{iP_xk_x}{k}-P_z\right)e^{t(kC_z-ik_xC_x)}. \tag{5.6}$$

Due to the exponential time behavior of $\Phi_{\mathbf{k}}(t)$, it is easy to obtain the expressions for $\psi_{\mathbf{k}}(t)$ and $\eta_{\mathbf{k}}(t)$. Introducing the definition

$$\lambda_{\mathbf{k}}=kC_z-ik_xC_x, \tag{5.7}$$

we can represent the answer in the following form:

$$\psi_{\mathbf{k}}(t)=\left(\frac{P}{Ck}\right)\frac{\lambda_{\mathbf{k}}^3}{gk+\lambda_{\mathbf{k}}^2}e^{\lambda_{\mathbf{k}}t}, \tag{5.8}$$

$$\eta_{\mathbf{k}}(t)=\left(\frac{P}{C}\right)\frac{\lambda_{\mathbf{k}}^2}{gk+\lambda_{\mathbf{k}}^2}e^{\lambda_{\mathbf{k}}t}. \tag{5.9}$$

The radiated surface waves influence the motion of the vortex ring. The terms produced by the field $\eta_{\mathbf{k}}(t)$ in the equations of motion for the ring come from the part $\int\dot{\eta}\Phi d^2\mathbf{r}_\perp$ in the Lagrangian (3.13). Using Eq. (5.6) for the Fourier transform of Φ, we can represent these terms as follows:

$$\delta\dot{X}=\int\frac{d^2\mathbf{k}}{(2\pi)^2}\dot{\eta}_{\mathbf{k}}\frac{ik_x}{k}e^{kZ+ik_xX},$$

$$\delta\dot{Z}=\int\frac{d^2\mathbf{k}}{(2\pi)^2}\dot{\eta}_{\mathbf{k}}e^{kZ+ik_xX},$$

$$\delta\dot{P}_x=-\int\frac{d^2\mathbf{k}}{(2\pi)^2}\dot{\eta}_{\mathbf{k}}(ik_x)\left(P_z+\frac{iP_xk_x}{k}\right)e^{kZ+ik_xX},$$

$$\delta\dot{P}_z=-\int\frac{d^2\mathbf{k}}{(2\pi)^2}\dot{\eta}_{\mathbf{k}}k\left(P_z+\frac{iP_xk_x}{k}\right)e^{kZ+ik_xX}.$$

We can use Eq. (5.9) to obtain the nonconservative corrections for time derivatives of the ring parameters from these expressions. It is convenient to write down these corrections in the autonomic form

$$\delta\dot{X}=\left(\frac{P}{C}\right)\int\frac{d^2\mathbf{k}}{(2\pi)^2}\left(\frac{ik_x}{k}\right)\frac{(kC_z-ik_xC_x)^3}{gk+(kC_z-ik_xC_x)^2}e^{-2k|Z|}, \tag{5.10}$$

$$\delta\dot{Z}=\left(\frac{P}{C}\right)\int\frac{d^2\mathbf{k}}{(2\pi)^2}\frac{(kC_z-ik_xC_x)^3}{gk+(kC_z-ik_xC_x)^2}e^{-2k|Z|},\tag{5.11}$$

$$\delta\dot{P}_x=-\left(\frac{P}{C}\right)^2\int\frac{d^2\mathbf{k}}{(2\pi)^2}\left(\frac{ik_x}{k}\right)\times\frac{(kC_z-ik_xC_x)^2(C_z^2k^2+C_x^2k_x^2)}{gk+(kC_z-ik_xC_x)^2}e^{-2k|Z|},\tag{5.12}$$

$$\delta\dot{P}_z=-\left(\frac{P}{C}\right)^2\int\frac{d^2\mathbf{k}}{(2\pi)^2}\times\frac{(kC_z-ik_xC_x)^2(C_z^2k^2+C_x^2k_x^2)}{gk+(kC_z-ik_xC_x)^2}e^{-2k|Z|},\tag{5.13}$$

where C_x and C_z can be understood as explicit functions of $\mathbf{P}$ defined by the dependence $\mathbf{C}(\mathbf{P})=\partial\mathcal{E}/\partial\mathbf{P}$. More exact definition of C_x and C_z as $\dot{X}$ and $\dot{Z}$ is not necessary.

To analyze the above integrals, let us first perform there the integration over the angle φ in $\mathbf{k}$ space. It is convenient to use the theory of contour integrals in the complex plane of variable $w=\cos\varphi$. The contour γ of integration in our case goes clockwise just around the cut, which is from -1 to $+1$. We define the sign of the square root $R(w)=\sqrt{1-w^2}$ so that its values are positive on the top side of the cut and negative on the bottom side. After introducing the quantities

$$a=\frac{C_z}{C_x},\quad \omega_{\mathbf{k}}^2=gk,\quad b_{\mathbf{k}}=\frac{\omega_{\mathbf{k}}}{C_xk}=\frac{1}{C_x}\sqrt{\frac{g}{k}},\tag{5.14}$$

we have to use the following relations:

$$I_1(a,b)\equiv-\oint_\gamma\frac{dw}{\sqrt{1-w^2}}\frac{w(w+ia)^3}{b^2-(w+ia)^2}=\pi(1+2b^2)+\pi i\left(\frac{(b-ia)b^2}{\sqrt{1-(b-ia)^2}}-\text{c.c.}\right),\tag{5.15}$$

$$I_2(a,b)\equiv i\oint_\gamma\frac{dw}{\sqrt{1-w^2}}\frac{(w+ia)^3}{b^2-(w+ia)^2}=2\pi a+\left(\frac{b^2}{\sqrt{1-(b-ia)^2}}+\text{c.c.}\right),\tag{5.16}$$

$$J_1(a,b)\equiv i\oint_\gamma\frac{dw}{\sqrt{1-w^2}}\frac{w(w+ia)^2(w^2+a^2)}{b^2-(w+ia)^2}=-4\pi ab^2+\pi\left(\frac{b(b-ia)(a^2+(b-ia)^2)}{\sqrt{1-(b-ia)^2}}+\text{c.c.}\right),\tag{5.17}$$

$$J_2(a,b)\equiv\oint_\gamma\frac{dw}{\sqrt{1-w^2}}\frac{(w+ia)^2(w^2+a^2)}{b^2-(w+ia)^2}=-2\pi(a^2+b^2+1/2)-\pi i\left(\frac{b[a^2+(b-ia)^2]}{\sqrt{1-(b-ia)^2}}-\text{c.c.}\right),\tag{5.18}$$

where the sign of the complex square root should be taken in accordance with the previous choice. It can easily be seen that the integrals I_2 and J_1 have resonance structure at $a\ll1$ and $|b|<1$. This is the Cherenkov effect itself. Now the expressions (5.10)–(5.13) take the form

$$\delta\dot{X}=\frac{P_x}{(2\pi)^2}\int_0^{+\infty}I_1(a,b_{\mathbf{k}})k^2e^{-2k|Z|}dk=\frac{P_x}{(2\pi)^2}\left(\frac{g}{C_x^2}\right)^3F_1\left(a,\frac{2g|Z|}{C_x^2}\right),\tag{5.19}$$

$$\delta\dot{Z}=\frac{P_x}{(2\pi)^2}\int_0^{+\infty}I_2(a,b_{\mathbf{k}})k^2e^{-2k|Z|}dk=\frac{P_x}{(2\pi)^2}\left(\frac{g}{C_x^2}\right)^3F_2\left(a,\frac{2g|Z|}{C_x^2}\right),\tag{5.20}$$

$$\delta\dot{P}_x=\frac{P_x^2}{(2\pi)^2}\int_0^{+\infty}J_1(a,b_{\mathbf{k}})k^3e^{-2k|Z|}dk=\frac{P_x^2}{(2\pi)^2}\left(\frac{g}{C_x^2}\right)^4G_1\left(a,\frac{2g|Z|}{C_x^2}\right),\tag{5.21}$$

$$\delta\dot{P}_z=\frac{P_x^2}{(2\pi)^2}\int_0^{+\infty}J_2(a,b_{\mathbf{k}})k^3e^{-2k|Z|}dk=\frac{P_x^2}{(2\pi)^2}\left(\frac{g}{C_x^2}\right)^4G_2\left(a,\frac{2g|Z|}{C_x^2}\right).\tag{5.22}$$

Here the functions $F_1(a,Q)\cdots G_2(a,Q)$ are defined by the integrals

$$F_1(a,Q)=\int_0^{+\infty}I_1\left(a,\frac{1}{\sqrt{\xi}}\right)\exp(-Q\xi)\xi^2d\xi,\tag{5.23}$$

$$F_2(a,Q)=\int_0^{+\infty}I_2\left(a,\frac{1}{\sqrt{\xi}}\right)\exp(-Q\xi)\xi^2d\xi,\tag{5.24}$$

$$G_1(a,Q)=\int_0^{+\infty}J_1\left(a,\frac{1}{\sqrt{\xi}}\right)\exp(-Q\xi)\xi^3d\xi,\tag{5.25}$$

$$G_2(a,Q)=\int_0^{+\infty}J_2\left(a,\frac{1}{\sqrt{\xi}}\right)\exp(-Q\xi)\xi^3d\xi,\tag{5.26}$$

and $Q=2g|Z|/C_x^2$ is a dimensionless quantity. [If we consider a fluid with surface tension σ, then two parameters appear: Q and $T=g\sigma/C_x^4$. In that case one should substitute

$b_{\mathbf{k}} \to \sqrt{1/\xi + T\xi}$ as the second argument of the functions I_1, I_2, J_1, J_2 in the integrals (5.23)–(5.26).] The Cherenkov effect is most clear when the motion of the ring is almost horizontal. In this case $a \to +0$, and it is convenient to rewrite these integrals without use of complex functions,

$$F_1(+0,Q) = \pi \int_0^{+\infty} (\xi^2 + 2\xi)\exp(-Q\xi)\,d\xi - 2\pi \times \int_0^1 \frac{\xi\,d\xi}{\sqrt{1-\xi}}\exp(-Q\xi), \tag{5.27}$$

$$F_2(+0,Q) = G_1(+0,Q) = -2\pi \int_1^{+\infty} \frac{\xi^{3/2}\,d\xi}{\sqrt{\xi - 1}}\exp(-Q\xi), \tag{5.28}$$

$$G_2(+0,Q) = -\pi \int_0^{+\infty} (\xi^3 + 2\xi^2)\exp(-Q\xi)\,d\xi + 2\pi \int_0^1 \frac{\xi^2\,d\xi}{\sqrt{1-\xi}}\exp(-Q\xi). \tag{5.29}$$

Here the square root is the usual positive defined real function. We see that only resonant wave numbers contribute to the functions F_2 and G_1, while F_1 and G_2 are determined also by small values of ξ which correspond to the large-scale surface deviation comoving with the ring. So the effect of the Cherenkov radiation on the vortex ring motion is the most distinct in the equations for $\dot{Z}$ and $\dot{P}_x$. It is especially important for P_x because the radiation of surface waves is the only reason for change of this quantity in the frame of our approximation.

The typical values of Q are large in practical situations. In this limit, asymptotic values of the above integrals are

$$F_1(+0,Q) \approx -\frac{9\pi}{2Q^4}, \quad G_2(+0,Q) \approx \frac{18\pi}{Q^5},$$

$$F_2(+0,Q) = G_1(+0,Q) \approx -2\pi\sqrt{\pi}\frac{\exp(-Q)}{\sqrt{Q}},$$

and

$$\delta\dot{X} \approx -\frac{9}{64\pi}\frac{P}{|Z|^3}\frac{1}{Q},$$

$$\delta\dot{Z} \approx -\frac{1}{16\sqrt{\pi}}\frac{P}{|Z|^3}Q^{2+1/2}\exp(-Q),$$

$$\delta\dot{P}_x \approx -\frac{1}{32\sqrt{\pi}}\frac{P^2}{|Z|^4}Q^{3+1/2}\exp(-Q),$$

$$\delta\dot{P}_z \approx +\frac{9}{32\pi}\frac{P^2}{|Z|^4}\frac{1}{Q}.$$

It follows from these expressions that the interaction with the surface waves is small in comparison with the interaction between the ring and its image, if $Q \gg 1$. The corresponding small factors are $1/Q$ for X and P_z, and $Q^{2+1/2}\exp(-Q)$ for Z. Contrary to the flat boundary, now P_x is not conserved. It decreases exponentially slowly and this is the main effect of Cherenkov radiation.

We see also that the interaction with waves turns the vector **P** towards the surface, which results in a faster boundary approach by the ring track.

VI. CONCLUSIONS

In this paper, we have derived the simplified Lagrangian for the description of the motion of deep vortex rings under the free surface of perfect fluid. We have analyzed the integrable dynamics corresponding to the pure interaction of the single point vortex ring with its image. It was found that there are three types of qualitatively different behavior of the ring. The interaction of the ring with the surface has an attractive character in all three regimes. The Fourier components of radiated Cherenkov waves were calculated for the case when the vortex ring comes from infinity and has both horizontal and vertical components of the velocity. The nonconservative corrections to the equations of motion of the ring, due to Cherenkov radiation, were derived. Due to these corrections, the track of the ring bends towards the surface faster than in the case of a flat surface. For simplicity, all calculations in Sec. V were performed for a single ring. The generalization for the case of many rings is straightforward.

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